The Penrose type twistor correspondence for the exceptional simple Lie group G_2

Fuminori Nakata

Faculty of Human Development and Culture, Fukushima University

1 Introduction

The following diagram is known:

$$\begin{tikzcd}
& G_2/U(2)_- \arrow[dl, "\varpi"', "\mathbb{C}P^2"] \arrow[dr, "\pi_-", "S^2"'] & & G_2/U(2)_+ \arrow[dl, "\pi_+"', "S^2"] \\
G_2/SU(3) & & G_2/SO(4) &
\end{tikzcd} \tag{1.1}$$

Here, $U(2)_\pm$ are two types of $U(2)$ embedded in G_2. As well known, $G_2/SU(3)$ is isomorphic to S^6, and S^6 is equipped with a natural non-integrable almost complex structure. It is also well known that $G_2/SO(4)$ is a 8-dimensional Riemannian symmetric space equipped with a quaternion Kähler structure. The fibration $\pi_+ : G_2/U(2)_+ \to G_2/SO(4)$ is the *twistor fibration* of the quaternion Kähler structure. The map $\varpi : G_2/U(2)_- \to G_2/SU(3)$ is also known as a *twistor fibration* with respect to the almost complex structure on S^6.

On the other hand, on the diagram (1.1), the double fibration given by ϖ and π_- is considered as the "Penrose type" twistor correspondence which is summarized as follows. Let Z be a complex 3-fold. This Z is called the *twistor space*. If Z contains a rational curve Y with normal bundle holomorphically isomorphich to $\mathcal{O}(1) \oplus \mathcal{O}(1)$, such rational curve is called *twistor line*. In general, the set of twistor lines consists a complex 4-fold M with naturally defined self-dual complex conformal structure. This M is called the *space-*

time. Then we obtain the following double fibration:

$$\begin{tikzcd} & F \arrow[dl, "\varpi"'] \arrow[dr, "\pi"] & \\ Z^3 & & M^4 \end{tikzcd} \tag{1.2}$$

For each $p \in Z$, the set $\pi(\varpi^{-1}(p))$ is 2-dimensional complex submanifold on M in general. Such complex surfaces are called β-surfaces, and the family of β-surfaces characterizes the self-dual structure of M.

In this article, we show that the double fibration by ϖ and π_- on the diagram (1.1) actually have an analogous structure with the Penrose's double fibraion. We show that for each $p \in S^6 \simeq G_2/SU(3)$, the subset $\mathfrak{S}_p = \pi_-(\varpi(p))$ is a totally geodesic, totally quaternionic 4-dimensional submanifold on $G_2/SO(4)$ (Theorem 6.3). Further, we show that there exists a symmetric 3-form γ, which satisfies certain integrable condition (Theorem 6.4). In the way to prove these theorem, we study the detail structure of the symmetric space $G_2/SO(4)$, for example, we describe explicitly the tangent space.

Here we remark about the recent work given by Enoyoshi-Tsukada [4]. They notice to the following another double fibration

$$\begin{tikzcd} & G_2/SO(3) \arrow[dl] \arrow[dr] & \\ G_2/SU(3) & & G_2/SO(4) \end{tikzcd} \tag{1.3}$$

This double fibration is related to the *special Lagrangian submanifold* (or *totally real submanifold*) of S^6. The idea of Penrose type twistor correspondence also takes an important role of this theory. We, however, do not investigate in this theory in this article.

2 Construction of the fibration

2.1 quaternion and G_2

Let $\mathbb{H}$ be the quaterenions generated by $\{1, i, j, k\}$ where $i^2 = j^2 = k^2 = -1$ and $k = ij = -ji$. We write $Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\}$. Let

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon = \mathrm{Span}_{\mathbb{R}}\langle 1, i, j, k, i\varepsilon, j\varepsilon, k\varepsilon\rangle = \mathbb{R} \oplus \mathrm{Im}\,\mathbb{O} \tag{2.1}$$

be the Cayley numbers. The multiplication on $\mathbb{O}$ is defined by $(a + b\varepsilon)(c + d\varepsilon) = (ac - \bar{d}b) + (da + b\bar{c})\varepsilon$. The inner product on $\mathbb{O}$ is $\langle x, y\rangle = \mathrm{Re}(x\bar{y})$. The 14-dimensional compact Lie group G_2 is defined as the aoutomorphism group of $\mathbb{O}$, that is

$$G_2 = \{g \in GL(\mathbb{O}) \,|\, g(xy) = g(x)g(y) \text{ for any } x, y \in \mathbb{O}\}. \tag{2.2}$$

Its Lie algebra $\mathfrak{g}_2$ is given by

$$\mathfrak{g}_2 = \{X \in \mathrm{End}(\mathbb{O}) \,|\, X(xy) = X(x)y + xX(y) \text{ for any } x, y \in \mathbb{O}\}. \tag{2.3}$$

As well known, $G_2 \subset SO(\mathrm{Im}\,\mathbb{O}) \simeq SO(7)$ and consequently $\mathfrak{g}_2 \subset \mathfrak{so}(7)$. We define an inner product on $\mathfrak{g}_2$ by

$$\langle X, Y\rangle = -\mathrm{Tr}\, XY \qquad (X, Y \in \mathfrak{g}_2). \tag{2.4}$$

2.2 almost complex structure on S^6

Let $S^6 = \{p \in \mathrm{Im}\,\mathbb{O} \mid |p| = 1\}$ be the set of *imaginary units*. The tangent space at $p \in S^6$ is $T_pS^6 = \{u \in \mathrm{Im}\,\mathbb{O} \mid \langle u, p\rangle = 0\}$. A natural almost complex structure J on S^6 is defined by

$$J_p : T_pS^6 \to T_pS^6, \qquad J_p(u) = pu. \tag{2.5}$$

It is well-known that the almost complex structure J is not integrable.

The group G_2 acts transitively on S^6 and the isotropy subgroup at $i \in S^6$ is $SU(3)$ (see [5]). Hence $S^6 \simeq G_2/SU(3)$.

2.3 associative Grassmannian

A 3-dimensional subspace $V \subset \operatorname{Im}\mathbb{O}$ is called an *associative 3-plane* if and only if $(xy)z = x(yz)$ holds for any $x, y, z \in V$. We put

$$\mathbb{H}_V = \mathbb{R} \oplus V. \tag{2.6}$$

Then the 3-plane V is associative if and only if $\mathbb{H}_V \subset \mathbb{O}$ is a quaternion subspace, i.e. $\mathbb{H}_V$ is a subalgebra of $\mathbb{O}$ and is isomorphic to $\mathbb{H}$.

Let $Gr_3^+(\operatorname{Im}\mathbb{O})$ be the Grassmann manifold of oriented 3-planes on $\operatorname{Im}\mathbb{O}$. We write

$$Gr_{\text{ass}}^+(\operatorname{Im}\mathbb{O}) = \{V \in Gr_3^+(\operatorname{Im}\mathbb{O}) \mid V \text{ is associative}\}, \tag{2.7}$$

and we call $Gr_{\text{ass}}^+(\operatorname{Im}\mathbb{O})$ as *associative Grassmannian.* The following properties hold (see [5]).

Proposition 2.1. (i) *If $x, y \in \operatorname{Im}\mathbb{O}$ and $x \perp y$, then $\{x, y, xy\}$ spans an associative 3-plane. Any associative 3-plane is written in this way. Consequently, any associative 3-plane has a natural orientation.*
(ii) *G_2 acts transitively on $Gr_{\text{ass}}^+(\operatorname{Im}\mathbb{O})$. The isotropy subgroup at* $\operatorname{Im}\mathbb{H}$ *is $SO(4)$. Hence $Gr_{\text{ass}}^+(\operatorname{Im}\mathbb{O}) \simeq G_2/SO(4)$ and $Gr_{\text{ass}}^+(\operatorname{Im}\mathbb{O})$ is an 8-dimensional Riemannian symmetric space.*

Further, $Gr_{\text{ass}}^+(\operatorname{Im}\mathbb{O}) \simeq G_2/SO(4)$ has a *quaternion Kähler structure* which we will explain in Section 5 (see also [2]). We also describe the isotropy subgroup $SO(4) \subset G_2$ explicitly in section 3.

2.4 associative calibration

The *associative calibration* φ is the 3-linear form on $\operatorname{Im}\mathbb{O}$ defined by

$$\varphi(x, y, z) = \langle x, yz\rangle. \tag{2.8}$$

The following is known.

Proposition 2.2 ([5])**.** (i) *Let $V \in Gr_3^+(\operatorname{Im}\mathbb{O})$ and $\{v_1, v_2, v_3\}$ is an oriented orthonormal basis on V. Then*

$$\varphi(V) = \varphi(v_1, v_2, v_3) \tag{2.9}$$

is independent of the choice of the basis.
(ii) $\varphi(\overline{V}) = -\varphi(V)$, *where* $\overline{V}$ *is the orientation reversing of* V.
(iii) $|\varphi(V)| \leq 1$. *In particular* $\varphi(V) = 1$ *if and only if* V *is associative.*

Consequently, we can write

$$Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) = \{V \in Gr^+_3(\mathrm{Im}\,\mathbb{O}) \mid \varphi(V) = 1\}. \tag{2.10}$$

2.5 flag manifold $F^+_{1,\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$

We have the following double fibration

$$\begin{tikzcd} & Gr^+_2(\mathrm{Im}\,\mathbb{O}) \arrow[dl, "\varpi"'] \arrow[dr, "\pi_-"] & \\ S^6 & & Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}), \end{tikzcd} \tag{2.11}$$

where ϖ and π_- is defined as follows: let $\xi \in Gr^+_2(\mathrm{Im}\,\mathbb{O})$ and $\{v_1, v_2\}$ be an oriented orthonormal basis of ξ, then

$$\varpi(\xi) = v_1v_2 \in S^6, \qquad \pi_-(\xi) = \mathrm{Span}_{\mathbb{R}}\langle v_1, v_2, v_1v_2\rangle \in Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}). \tag{2.12}$$

The oriented 2-plane $V = \{v_1, v_2\}$ is one-to-one corresponds with the pair $(p, V) \in S^6 \times Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ satisfying $p \in V$ so that $p = v_1v_2$ and $V = \mathrm{Span}_{\mathbb{R}}\langle v_1, v_2, v_1v_2\rangle$. Hence the Grassmann manifold $Gr^+_2(\mathrm{Im}\,\mathbb{O})$ is naturally identified with the flag manifold

$$Fl^+_{1,\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) = \{(p, V) \in S^6 \times Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \mid p \in V\}. \tag{2.13}$$

Hence we can replace (2.11) by

$$\begin{tikzcd} & Fl^+_{1,\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \arrow[dl, "\varpi"'] \arrow[dr, "\pi_-"] & \\ S^6 & & Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}), \end{tikzcd} \tag{2.14}$$

In this notation, $\varpi(p, V) = p, \pi_-(p, V) = V$ are the natural projections.

The group G_2 acts $Fl^+_{1,\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ transitively, and the isotorpy subgroup at $(i, \mathrm{Im}\,\mathbb{H})$ is

$$U(2)_- = SU(3) \cap SO(4) = \{g \in G_2 \mid g(i) = i, g(\mathrm{Im}\,\mathbb{H}) = \mathrm{Im}\,\mathbb{H}\}. \tag{2.15}$$

This group is isomorphic to $U(2)$, which we see in the next section. In this way we obtain

$$Gr_2^+(\mathrm{Im}\,\mathbb{O}) \simeq Fl_{1,\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O}) \simeq G_2/U(2)_-. \tag{2.16}$$

2.6 submanifolds in S^6 and $Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O})$

The following proposition means π_- is a $\mathbb{CP}^1$-bundle, while ϖ is a $\mathbb{CP}^2$-bundle.

Proposition 2.3. (i) *For each* $V \in Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O})$, $Y_V = \varpi(\pi^{-1}(V))$ *is a puedo-holomorphic* $\mathbb{CP}^1$ *in* S^6.
(ii) *For each* $p \in S^6$, $\mathfrak{S}_p = \pi(\varpi^{-1}(p))$ *has a natural complex structure and is biholomorphic to* $\mathbb{CP}^2$.

Proof. We have $Y_V = \{p \in V \mid |p| = 1\} = S^6 \cap V \simeq S^2$. For each $p \in Y_V$, we can write $V = \mathrm{Span}_{\mathbb{R}}\langle p, x, J_p x\rangle$ for some $x \in T_pS^6$. Then $T_pY_V = \mathrm{Span}_{\mathbb{R}}\langle x, J_p x\rangle$ is a complex line in $T_pS^6 \simeq \mathbb{C}^3$. Thus Y_V is a psuedo-complex $\mathbb{CP}^1$ in S^6. So (i) is proved.

Next, for $p \in S^6$, we have

$$\mathfrak{S}_p = \{V \in Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O}) \mid p \in V\}.$$

When $p \in V \in Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O})$, we can write $V = \mathrm{Span}_{\mathbb{R}}\langle p, x, J_p x\rangle$ for some $x \in T_pS^6$. Such V one-to-one corresponds with the complex line $\mathrm{Span}_{\mathbb{R}}\langle x, J_p x\rangle \subset T_pS^6 \simeq \mathbb{C}^3$. Hence $\varpi^{-1}(p)$ is naturally identified with the complex projectivization of $T_pS^6 \simeq \mathbb{C}^3$. $\square$

3 Explicit description of the subgroups

3.1 $SO(4) \subset G_2$

For $(q_1, q_2) \in Sp(1) \times Sp(1)$, we define

$$\rho(q_1, q_2)(a + b\varepsilon) = q_1 a \bar{q}_1 + (q_2 b \bar{q}_1)\varepsilon \qquad (a \in \mathrm{Im}\,\mathbb{H}, b \in \mathbb{H}).$$

It is known that ρ defines an homomorphism $Sp(1) \times Sp(1) \to G_2$. In a matrix style, we can write

$$\rho(q_1, q_2) = \begin{pmatrix} \mathrm{Ad}_{q_1} & O \\ O & L_{q_2} R_{\bar{q}_1} \end{pmatrix} \tag{3.1}$$

with respect to the decomposition $\mathrm{Im}\,\mathbb{O} \simeq \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$. Since the kernel of ρ is $\mathbb{Z}_2 \simeq \{\pm(1,1)\}$, ρ defines an embedding $SO(4) \simeq (Sp(1) \times Sp(1))/\mathbb{Z}_2 \to G_2$. Further, we have the following (see [5])

$$SO(4) = \left\{ \begin{pmatrix} * & O \\ O & * \end{pmatrix} \in G_2 \right\} = \{ g \in G_2 \mid g(\mathrm{Im}\,\mathbb{H}) = \mathrm{Im}\,\mathbb{H} \} \tag{3.2}$$

3.2 $U(2)_\pm$ and $SU(3)$

Two subgroups of G_2 are defined by

$$U(2)_+ = \rho(Sp(1) \times U(1)), \qquad U(2)_- = \rho(U(1) \times Sp(1)), \tag{3.3}$$

where $U(1) = \{ q \in \mathbb{C} \subset \mathbb{H} \mid |q| = 1 \} \subset Sp(1)$. Though both subgroups are abstractly ismorphic to $U(2)$, the embeddings are not equivalent to each other. Actually, for example, the homotopy types of $G_2/U(2)_\pm$ are different (see [7]).

Another subgroup is defined by

$$SU(3) = \{ g \in G_2 \mid g(i) = i \}. \tag{3.4}$$

The subgroups $SO(4)$, $U(2)_-$, $SU(3)$ are simply characterized by the block decomposition of 7×7 matrices, and we easily see $U(2)_- = SU(3) \cap SO(4)$.

4 Twistor correspondence

We compare our double fibration (2.14) with the Penrose's twistor correspondence.

4.1 The idea of Penrose's twistor correspondence

Penrose's theory ([8]) concerns with the correspondence between a complex 3-fold Z (called the *twistor space*) and a self-dual complex 4-fold M (called the *space-time*). The correspondence is constructed in the following way.

Let Z be a complex 3-fold. We notice to the family *twistor lines* $\{Y_t\}_{t\in M}$, that is, the family of rational curves (i.e. $Y_t \simeq \mathbb{CP}^1$) in Z such that the normal bundle N is biholomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. By the deformation theory, such family is parametrized by a complex 4-fold M. If we put $F = \{(z,t) \in Z \times M \mid z \in Y_t\}$, we obtain the double fibration

$$\begin{tikzcd} & F \arrow[dl, "\varpi"'] \arrow[dr, "\pi"] \arrow[d, phantom, "\mathbb{CP}^1"] & \\ Z & & M \end{tikzcd} \tag{4.1}$$

where ϖ and π are natural projection.

For each $t \in M$, the corresponding object in Z is by definition $\varpi(\pi^{-1}(t)) = Y_t$, which is a holomorphic $\mathbb{CP}^1$ in Z.

On the other hand, for each $z \in Z$, the corresponding object in M is $\mathfrak{S}_z = \pi(\varpi^{-1}(z))$. Each $\mathfrak{S}_z$ is, if not empty, a 2-dimensional complex submanifold in M and is called *β-surface*. There is a unique complex conformal structure $[g]$ on M satisfying $g|_{\mathfrak{S}_z} = 0$ for any $z \in Z$. We can prove that this conformal structure $[g]$ is *self-dual* (i.e. half conformally flat).

4.2 Twistor correspondence for $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$

Our double fibration (2.14) is quite similar to the Penrose's double fibration (4.1) in the following sense.

The correspondence spaces F and $Fl^+_{1,\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ are both the total space of $\mathbb{CP}^1$-bundle over the "space-time" M and $Gr^+_{\mathrm{ass}}(\mathrm{Im}(\mathbb{O})$.

The twistor space Z is a complex 3-fold while S^6 is a real 6-dimensional manifold with an almost complex structure. Z has a family of twistor lines $\{Y_t\}$ ($Y_t \simeq \mathbb{CP}^1$) while S^6 has a family of psuedo holomorphic curves $\{Y_V\}$ ($Y_V \simeq \mathbb{CP}^1$).

The space-time M is a complex 4-fold while $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ is a real 8-dimensional quaternion Kähler manifold. M has a family of β-surfaces $\{\mathfrak{S}_z\}$ while $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ has a family of submanifolds $\{\mathfrak{S}_p\}$ ($\mathfrak{S}_p \simeq \mathbb{CP}^2$).

	Penrose's case	Our case
corresp. sp.	F $\mathbb{CP}^1$-bundle over M	$Fl^+_{1,\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ $\mathbb{CP}^1$-bundle over $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$
twistor space	Z (complex 3-fold) twistor lines $\{Y_t\}$	S^6 (almost complex 6-fold) psued-holo. curves $\{Y_V\}$
space-time	M (complex 4-fold) self-dual β-sufaces $\{\mathfrak{S}_z\}$	$Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ (q. Kähler 8-fold) ?? submanifolds $\{\mathfrak{S}_p\}$

In this comparison, it seems natural to expect that $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ has some extra geometric structure corresponding with the self-dual structure on M. We investigate this geometric structure in Section 5 and 6.

5 Explicit description of the tangent space

5.1 Tangent space of $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$

Proposition 5.1. *There is a natural identification*

$$T_oGr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \simeq \left\{ f \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\,\mathbb{H}, \mathbb{H}) \;\middle|\; f(i)i + f(j)j + f(k)k = 0 \right\}. \tag{5.1}$$

where $o = \mathrm{Im}\,\mathbb{H}$ is the base point on $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$.

Proof. We have $T_oGr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \simeq T_oG_2/SO(4) \simeq \mathfrak{g}_2/\mathfrak{so}(4) \simeq \mathfrak{p}$, where $\mathfrak{g}_2 = \mathfrak{so}(4) \oplus \mathfrak{p}$ is the Cartan decomposition for $G_2/SO(4)$. In the matrix style,

$$\mathfrak{so}(4) = \left\{ \begin{pmatrix} * & O \\ O & * \end{pmatrix} \in \mathfrak{g}_2 \right\}, \qquad \mathfrak{p} = \left\{ \begin{pmatrix} O & -f^* \\ f & O \end{pmatrix} \in \mathfrak{g}_2 \right\}.$$

So we check that $X = \begin{pmatrix} O & -f^* \\ f & O \end{pmatrix}$ $(f \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\,\mathbb{H}, \mathbb{H}))$ is contained in $\mathfrak{p}$ if and only if f satisfies the condition $f(i)i + f(j)j + f(k)k = 0$.

For each $x \in \mathrm{Im}\,\mathbb{H}$ we have $X(x) = f(x)\varepsilon$. On the other hand, for $x, y \in \mathrm{Im}\,\mathbb{H}$, we obtain

$$X(xy) = X(x)y + xX(y)$$

by the definition of $\mathfrak{g}_2$. Hence

$$f(xy)\varepsilon = (f(x)\varepsilon)y + x(f(y)\varepsilon) = (f(x)\bar{y})\varepsilon + (f(y)x)\varepsilon,$$

that is,

$$f(xy) = f(x)\bar{y} + f(y)x.$$

Putting $x = j, y = k$, we obtain $f(i)i + f(j)j + f(k)k = 0$. Thus

$$T_o Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \subset \left\{ f \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\,\mathbb{H}, \mathbb{H}) \;\middle|\; f(i)i + f(j)j + f(k)k = 0 \right\}.$$

Both vector spaces have real dimension 8, so these are equal. □

5.2 The quaternion Kähler structure on $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$

Let $V \in Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ and we define

$$\mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) = \left\{ f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{H}_V) \;\middle|\; f(e_1)e_1 + f(e_2)e_2 + f(e_3)e_3 = 0 \right\}, \tag{5.2}$$

where $\mathbb{H}_V = \mathbb{R} \oplus V$ is the quaternion subalgebra of $\mathbb{O}$ and $\{e_1, e_2, e_3\}$ is an oriented orthonormal basis of V. Then, as a consequence of (5.1), we obtain the identification

$$T_V Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \simeq \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V). \tag{5.3}$$

The vector space $\mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V)$ has a natural $\mathbb{H}_V$-module structrue defined by the left multiplication. This is the quaternion Kähler structure on $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$.

5.3 Infinitesimal deformation

A tangent vector $X \in T_V Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ is considered as an infinitesimal deformation of associative 3-plane in the following way.

For the simplicity, we assume $V = o = \mathrm{Im}\,\mathbb{H}$. Let $c(t)$ be a smooth curve on $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ satisfying $c(0) = o$. We can take a curve $g(t)$ on G_2 so that $c(t) = g(t) \cdot o$ and $g(0) = I$. Then the differential $g'(0)$ is determined uniquely up to $\mathfrak{so}(4)$. This means that the infinitesimal deformation $c'(0)$ can be written as

$$c'(0) = g'(0) + \mathfrak{so}(4) \quad \in \quad \mathfrak{g}_2/\mathfrak{so}(4). \tag{5.4}$$

5.4 The submanifold $\mathfrak{S}_p$

Lemma 5.2. *Let $p \in S^6$ and $V \in \mathfrak{S}_p$ (i.e. $p \in V \in Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$). Then*

$$T_V\mathfrak{S}_p = \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid f(p) = 0\}\,. \tag{5.5}$$

Proof. We assume $V = o = \mathrm{Im}\,\mathbb{H}$ for the simplicity. For a tangent vector $X \in T_o\mathfrak{S}_p$, let us take a smooth curve $c(t) = g(t)\cdot o$ on $\mathfrak{S}_p$ so that $g(t) \in G_2$, $g(0) = I$ and $c'(0) = X$.

By definition, $p \in g(t)\cdot o$ for any t. Changing the choice of $g(t)$ if needed, we can assume $g(t)\cdot p = p$. Then $g'(0)\cdot p = 0$. If $f \in \mathrm{Hom}_{\mathrm{ass}}(o, \mathbb{H})$ be the corresponding linear map with $X = c'(0) = g'(0) + \mathfrak{so}(4)$, we obtain $f(p) = 0$. □

Corollary 5.3. *Let $p \in S^6$. Then $\mathfrak{S}_p$ is a real 4-dimensional totally quaternionic submanifold of $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$.*

Proof. Direct calculation. □

6 The cone field and the symmetric 3-form

6.1 The cone field

In the Penrose's twistor theory, the self-dual structure (more precisely, the self-dual complex conformal structure) $[g]$ is defined so that its *null cone* is tangent to β-surfaces everywhere.

Similarly in our case, we notice to the *cone field* $\mathcal{C}$ defined by

$$\mathcal{C}_V := \bigcup_{V\in\mathfrak{S}_p} T_V\mathfrak{S}_p \qquad (V \in Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})). \tag{6.1}$$

Then

$$\begin{aligned}
\mathcal{C}_V &= \bigcup_{p\in S(V)} \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid f(p) = 0\} \\
&= \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid f(p) = 0 \text{ for some } p \in S(V)\} \\
&= \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid \mathrm{rank}_{\mathbb{R}}\, f < 2\} \\
&= \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid f(e_1) \times f(e_2) \times f(e_3) = 0\}
\end{aligned}$$

where $\{e_1, e_2, e_3\}$ is the oriented orthonormal basis of V and

$$x \times y \times z = \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x)) \tag{6.2}$$

is the *triple cross product*.

6.2 The symmetric 3-form

Let us define a *cubic form* $P : T_V Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \to \mathbb{H}_V$ by

$$P(f) = f(e_1) \times f(e_2) \times f(e_3) \tag{6.3}$$

which is independent of the choice of the oriented orthonormal basis $\{e_1, e_2, e_3\}$ on V. Since any polynomial one-to-one corresponds with a symmetric tensor, we can define $\mathbb{H}_V$-valued symmetric 3-form γ such that

$$P(f) = \gamma(f, f, f) \tag{6.4}$$

for any $f \in T_V Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$. By definition, we obtain

$$\mathcal{C}_V = \left\{ f \in T_V Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \mid \gamma(f, f, f) = 0 \right\}. \tag{6.5}$$

6.3 Main results

The associative Grassmannian $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \simeq G_2/SO(4)$ is equipped with the natural Riemannian metric h. Let ∇, R be the Riemannian connection and the Riemannian curvature tensor of h.

Theorem 6.1. *The symmetric 3-form γ is parallel, i.e. $\nabla\gamma = 0$.*

Proof. Let $\varrho : SO(4) \to SO(\mathfrak{p})$ be the isotropy representation of $G_2/SO(4)$ at the base point. Then by the property of the triple cross product, we obtain

$$P(\varrho(g)f) = g \cdot P(f). \tag{6.6}$$

Thus we obtain

$$\gamma(\varrho(g)\varphi, \varrho(g)\psi, \varrho(g)\chi) = g \cdot \gamma(\varphi, \psi, \chi). \tag{6.7}$$

Taking the differential, we obtain

$$\gamma(\varrho_*(A)\varphi,\psi,\chi)+\gamma(\varphi,\varrho_*(A)\psi,\chi)+\gamma(\varphi,\psi,\varrho_*(A)\chi)=A\cdot\gamma(\varphi,\psi,\chi). \quad (6.8)$$

for $A\in\mathfrak{so}(4)$. This means

$$\gamma(\nabla\varphi,\psi,\chi)+\gamma(\varphi,\nabla\psi,\chi)+\gamma(\varphi,\psi,\nabla\chi)=\nabla\gamma(\varphi,\psi,\chi) \quad (6.9)$$

i.e. ∇ is parallel. □

Lemma 6.2. *Let $p\in S^6$ and $V\in\mathfrak{S}_p$.*
(i) $\gamma(\varphi,\psi,\chi)=0$ *for any $\varphi,\psi,\chi\in T_V\mathfrak{S}_p$.*
(ii) *Let φ,ψ be the complex basis of $\mathfrak{S}_p\simeq\mathbb{CP}^2$. Then $\chi\in T_V\mathfrak{S}_p$ if and only if $\gamma(\chi,\varphi,\psi)=0$.*

Proof. This is directly checked when $V=\mathrm{Im}\mathbb{H}$ and $p=i$. Then the statement follows by the G_2-symmetricty. □

Theorem 6.3. *For any $p\in S^6$, the submanifold $\mathfrak{S}_p$ is real 4-dimensional, totally quaternionic and totally geodesic.*

Proof. By Corollary 5.3, we only need to show $\mathfrak{S}_p$ is totally geodesic.

For vector fields $v,w\in\mathfrak{X}(\mathfrak{S}_p)$, we have $[v,w]\in\mathfrak{X}(\mathfrak{S}_p)$. By $\gamma(v,v,v)=0$, we obtain $0=\nabla_w\gamma(v,v,v)=3\gamma(\nabla_w v,v,v)$. Hence by $\gamma(v,v,w)=0$,

$$2\gamma(\nabla_v v,v,w)=-\gamma(v,v,\nabla_v w)=-\gamma(v,v,\nabla_w v+[v,w])=0.$$

By Lemma 6.2, if we take v,w to be the complex basis, $\nabla_v v\in\mathfrak{X}(\mathfrak{S}_p)$.

On the other hand, by $\gamma(v,w,w)=0$,

$$2\gamma(v,\nabla_v w,w)=-\gamma(\nabla_v v,w,w)=0.$$

Hence $\nabla_v w\in\mathfrak{X}(\mathfrak{S}_p)$. Thus $\mathfrak{S}_p$ is totally geodesic. □

Theorem 6.4. *Let $p\in S^6$ and $V\in\mathfrak{S}_p$. Then, for any tangent vectors $\varphi,\psi\in T_V\mathfrak{S}_p$,*

$$\gamma(R(\varphi,\psi)\varphi,\varphi,\psi)=0. \quad (6.10)$$

Proof. We can assume $\{\varphi, \psi\}$ is the complex basis. Extending φ, ψ to a vector field, we obtain

$$R(\varphi, \psi)\varphi = \nabla_\varphi \nabla_\psi \varphi - \nabla_\psi \nabla_\varphi \varphi - \nabla_{[\varphi,\psi]}\varphi \quad \in \quad \mathfrak{X}(\mathfrak{S}_p). \tag{6.11}$$

Hence we obtain (6.10). □

Remark 6.5. Theorem 6.4 is an analogy of the self-duality. Actually, a Riemannian manifold (M, g) is self-dual if and only if

$$g(R(X, Y)X, Y) = 0$$

for any tangent vector X, Y (see [6]).

Acknowledgement

The author would like to thank Professor Hideya Hashimoto, Professor Misa Ohashi and Professor Katsuya Mashimo for taking part in meaningful arguments and giving useful advices. This work was partially supported by JSPS KAKENHI Grant Number 16K05118.

References

[1] M. F. Atiyah, N. Hitchin, I. M. Singer: *Self-duality in Four-dimensional Riemannian Geometry*, Proc. R. Soc. Lond. **A.362** (1978) 425-461

[2] L. Besse, *Einstein manifolds*, Springer (1987).

[3] R. L. Bryant, *Submanifolds and special structures on the octonions*, J. Diff. Geom. **17** (1982), 185-232.

[4] K. Enoyoshi, K. Tsukada, *Lagrangian submanifolds of S^6 and the associative Grassmann manifold*, preprint

[5] R. Harvey, H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47-157.

[6] C. LeBrun, L. J. Mason: *Nonlinear Gravitons, Null Geodesics, and Holomorphic Disks*, Duke Math. J. **136**, no.2 (2007) 205-273.

[7] F. Nakata: *Homotopy groups of* $G_2/Sp(1)$ *and* $G_2/U(2)$ Contemporary Perspectives in Differential Geometry and its Related Fields, Proceedings of the 5th International Colloquium on Differential Geometry and its Related Fields, (2018) 151-159.

[8] R. Penrose, *Nonlinear gravitons and curved twistor theory*, Gen. Rel. Grav. **7** (1976) 31-52.